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Stability of Three-Dimensional Boundary Layers

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A theory is developed for the linear stability of three-dimensional growing boundary layers. The method of multiple scales is used to derive partial-differential equations describing the temporal and spatial evolution of the complex amplitudes and wavenumbers of the disturbances. In general, these equations are elliptic, unless certain conditions are satisfied. For a monochromatic disturbance, these conditions demand that the ratio of the components of the complex group velocity be real, thereby relating the direction of growth of the disturbance to the disturbance wave angle. For a nongrowing boundary layer, this condition reduces to $d\alpha/d\beta$ being real, where α and β are the complex wavenumbers in the streamwise and crosswise directions, in agreement with the result obtained by using the saddle-point method. For a wavepacket, these conditions demand that the components of the complex group velocity be real. In all cases, the evolution equations are reduced to inhomogeneous ordinary-differential equations along real group velocity directions.

I. Introduction

IN this article, the linear three-dimensional stability of slightly nonparallel two- and three-dimensional flows are discussed. Two- and three-dimensional disturbances are possible in two-dimensional flows.

A. Two-Dimensional Disturbances of Two-Dimensional Parallel Flows

In the case of two-dimensional disturbances of two-dimensional parallel flows, one can study their linear behavior by assuming these disturbances to be traveling waves that can be expressed in dimensionless variables as

$$p(x,y,t) = \zeta(y)\exp[i(\alpha x - \omega t)] \tag{1}$$

where p is the pressue, t the time, x the distances along the body, and y the distances normal to the body. The disturbance is described by the parameters α and ω and by the amplitude function $\zeta(y)$. The function $\zeta(y)$ is governed by an eigenvalue problem that yields, for a given Reynolds R, a dispersion relation connecting α and ω in the form

$$\omega = \omega(\alpha) \tag{2}$$

In general, ω an α are complex and Eq. (2) provides two real relations connecting $\alpha_r, \alpha_i, \omega_r$, and ω_i, where the subscripts r and i refer to the real and imaginary parts of a given quantity. Thus, given two of these real parameters, one can determine the other two from Eq. (2). When α is assumed to be real and fixed, Eq. (2) provides a complex $\omega = \omega_r + i\alpha_i$ and the problem is called a temporal stability problem. When ω is assumed to be real and fixed, Eq. (2) provides a complex $\alpha = \alpha_r + i\alpha_i$ and the problem is called a spatial stability problem.

In the case of temporal stability, the growth rate of the disturbance is ω_i, while in the case of spatial stability, the growth rate of the disturbance is $-\alpha_i$. Gaster[1] proposed the transformation

$$\alpha_i = -\omega_i/\omega_r' \tag{3}$$

where $\omega_r' = d\omega_r/d\alpha$ is the real group velocity. This transformation works well when the imaginary part of the group velocity $\omega' = d\omega/d\alpha$ is small. When it is not small, Nayfeh and Padhye[2] proposed the transformations

$$\delta\alpha = -\omega_i/\omega' \tag{4a}$$

$$\delta\omega = -\alpha_i\omega' \tag{4b}$$

In Eq. (4a), ω' is complex and is determined from a temporal-stability calculation. Hence, $\delta\alpha$ is complex; its real part is the negative of the spatial growth rate, while its imaginary part is a correction to the wavenumber. In Eq. (4b), ω' is complex and is determined from a spatial-stability calculation. Hence, $\delta\omega$ is complex; its real part is the temporal growth rate, while its imaginary part is a correction to the frequency.

To calculate ω', one can determine two or more values of α and then use a finite-difference method. Alternatively, as described in Sec. V, one can use the solution of the adjoint homogeneous problem and express ω' in quadrature in terms of the eigenfunctions and their adjoints. Obviously, the second approach is more accurate than the first.

B. Three-Dimensional Disturbances of Two-Dimensional Parallel Flows

In the case of three-dimensional disturbances of two-dimensional parallel mean flows, these disturbances are expressed as

$$p(x,y,z,t) = \zeta(y)\exp[i(\alpha x + \beta z - \omega t)] \tag{5}$$

where z denotes distances normal to the flow direction (spanwise direction). For a given R, the eigenvalue problem provides a dispersion relation of the form

$$\omega = \omega(\alpha,\beta) \tag{6}$$

In general, ω, α, and β are complex. Thus, Eq. (6) provides two relations among the six real parameters, $\alpha_r, \alpha_i, \beta_r, \beta_i, \omega_r$, and ω_i. When four of these real parameters are specified, Eq. (6) provides the other two real parameters. In the case of temporal stability, α and β are assumed to be real and fixed, and Eq. (6) provides $\omega = \omega_r + i\omega_i$. Thus, the temporal growth rate is ω_i. In the case of spatial stability, ω is assumed to be

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real and fixed, and Eq. (6) provides two relations among the remaining four parameters. Then, the question is how to specify two other relations in order to determine the remaining four parameters. One can define a real wavenumber vector $\boldsymbol{k}$ whose magnitude k and direction ψ (wave angle) are defined by

$$k=(\alpha_r^2+\beta_r^2)^{1/2}, \quad \psi=\tan^{-1}(\beta_r/\alpha_r) \tag{7}$$

Similarly, one can define a real growth rate vector $\boldsymbol{\sigma}$ whose magnitude σ and direction $\bar{\psi}$ (growth direction) are defined by

$$\sigma=(\alpha_i^2+\beta_i^2)^{1/2}, \quad \bar{\psi}=\tan^{-1}(\beta_i/\alpha_i) \tag{8}$$

As will be shown in Secs. IV and V, ψ and $\bar{\psi}$ need not be equal.

For an incompressible flow, Squire[3] showed that the three-dimensional temporal stability of a two-dimensional parallel mean flow can be transformed into a two-dimensional temporal stability problem according to

$$k=(\alpha^2+\beta^2)^{1/2}, \quad \bar{R}=\alpha R/k, \quad \bar{\omega}=\omega k/\alpha \tag{9}$$

This transformation relates the wavenumbers α and β, the Reynolds number R, the frequency ω_r, and the growth rate ω_i of a three-dimensional wave to the wavenumber k, the Reynolds number $\bar{R}$, the frequency $\bar{\omega}_r$, and the growth rate $\bar{\omega}_i$ of a two-dimensional wave. Based on this transformation, Squire concluded that, in a two-dimensional incompressible boundary layer, the minimum critical Reynolds number is given by a two-dimensional wave. Moreover, the most unstable wave at a given Reynolds number is two dimensional so that it is necessary to consider the latter because disturbances which appear physically are two dimensional. However, the calculations of Mack[4] show that the most unstable wave at any frequency other than the most unstable one can well be three-dimensional.

C. Two-Dimensional Compressible Flows

For a compressible two-dimensional mean flow, Dunn and Lin[5] and Reshotko[6] showed that the three-dimensional stability equations cannot be transformed into two-dimensional equations, as in the incompressible case, because of the dissipation term in the energy equation. Mack[7] found that neglecting the dissipation term leads to at most a 10% error in the amplification rate. Moreover, as the Mach number M_e increases, three-dimensional disturbances become more and more important. When $M_e \approx 0.8$, the maximum amplification rate has a broad maximum as a function of the wave orientation. For supersonic flows, the numerical calculations of Mack[8] show that the most unstable wave is a three-dimensional wave. Moreover, during a numerical study, Mack[9] found that whenever the flow is supersonic, relative to the disturbance over some portion of the boundary-layer profile, there can be more than one unstable mode, in contrast with incompressible flows. The most unstable higher-order modes are two-dimensional, in contrast with the most unstable first mode, which is three dimensional. The effect of nonparallelism on the stability of compressible flows was analyzed by El-Hady and Nayfeh.[10] The nonparallel stability results are in better agreement with the Mach 2.2 experimental data of Laufer and Vrebalovich[11] and Kendall[12] than the parallel stability results.[13]

D. Three-Dimensional Incompressible Flows

From his flight tests on aircraft with sweptback wings, Gray[14] discovered that the boundary layer became turbulent closer to the leading edge than on a corresponding unswept wing. Using evaporation methods for indicating the state of the boundary layer, he discovered the existence of regularly spaced vortices whose axes lie in the streamwise direction. Since on a swept wing the spanwise pressure gradient deflects the boundary layer toward the region of low static pressure, the flow paths of the boundary-layer profiles differ from the potential flow streamlines and a crossflow develops in the direction normal to the streamlines (i.e., the mean flow is three dimensional). The crossflow profiles have inflection points, making them dynamically unstable and hence leading to the generation of the vortices.[15,16] This crossflow instability was confirmed in wind tunnels on large swept wings by Gregory and Walker.[15]

The feasibility of using suction to maintain laminar flow in the presence of crossflow instability was shown by Pfenninger et al.,[17] Bacon et al.,[18] Gault,[19] and Pfenninger and Bacon[20] on a 30-deg sweptback wing. This feasibility culminated in the successful maintenance of full-chord laminar flow on an X-21 wing.

The problem of laminar flow control on sweptback wings and the discovery of Gray[14] of the crossflow instability stimulated research into the linear stability of three-dimensional flows. In contrast with the problem of two-dimensional flows, disturbances in a three-dimensional flow are always three-dimensional. Stuart[15] derived the general linear equations that describe the three-dimensional stability of three-dimensional incompressible boundary layers over bodies, including the effects of boundary-layer growth and body and streamline curvatures. He showed that the temporal parallel stability problem of a three-dimensional incompressible flow can be reduced to that of a two-dimensional stability problem. For a given R and ω, his transformation relates the wavenumbers α and β in a three-dimensional flow with the streamwise and spanwise velocity components U and W to the wavenumber $k=(\alpha^2+\beta^2)^{1/2}$ in a two-dimensional flow having the velocity $\hat{U}=(\alpha U+\beta W)/k$. He used the inviscid form of these disturbance equations to solve a two-dimensional stability problem for the rotating disk boundary-layer profiles in the so-called critical direction. Brown[21] numerically solved the viscous eigenvalue problems for the flows over a rotating disk and a sweptback wing.

Since the uncoupling of the crosswise stability problem from the streamwise stability problem is artifical, a number of attempts have been directed toward determining the stability of the combined streamwise and spanwise stability problems to determine the suction requirements for maintaining laminar flow over sweptback wings. The form of the three-dimensional disturbance is given by Eq. (5) leading to the dispersion relation of Eq. (6) for a parallel mean flow. The temporal stability problem can easily be solved by using Stuart's transformation and solving a fourth-order system or by numerically integrating the original sixth-order system. The resulting temporal growth rates need to be converted to spatial growth rates, which can be used to determine the amplitude of the disturbance and hence the n factor for correlating the transition location.[22-24] Srokowski and Orszag[25] developed a computer code to calculate the maximum temporal amplification rate with respect to wavenumber for a given frequency from incompressible stability equations. This amplification rate is converted into a spatial amplification rate by using the real part of the group velocity, and is then integrated along a trajectory defined by the direction of the real part of the group velocity. Mack[26] performed spatial calculations for the rotating disk boundary layer by assigning the direction of growth to be the direction of the real part of the group velocity. Mack[4] performed numerical calculations for the Falkner-Skan-Cooke yawed-wedge boundary layers by assigning the direction of growth to be almost identical to the real part of the complex group-velocity angle.

It should be noted that the practical problem is laminar flow control over sweptback wings on which Gray discovered the crossflow instability. Since the mean flow over such wings is not easy to calculate, a number of investigators[4,15,21,26,27] studied alternate flows that exhibit the crossflow instability, such as the flow on a rotating disk and the Falkner-Skan-

Cooke yawed-wedge boundary layer. Although the latter flows exhibit the crossflow instability, it is not clear how one can quantitatively relate the stability results for such flows to those over sweptback wings. Moreover, for the rotating disk, the disturbance vortices take the form of equiangular spirals, whereas for the sweptback wing, the disturbance vortices have their axes in the streamwise direction. Thus, the stability of the actual three-dimensional flow over sweptback wings should be analyzed for designing laminar flow control.

E. Relation Between Temporal and Spatial Stabilities

Nayfeh and Padhye[2] used the method of multiple scales[28] to determine equations describing the modulations of the amplitude and phase of a three-dimensional wave propagating in a nonparallel three-dimensional incompressible boundary-layer flow. They used these equations to derive transformations relating temporal and spatial stabilities and a transformation relating the growth rates along two different directions. We alternatively derive these transformations from the dispersion relation given by Eq. (6). To this end, we apply the dispersion relation at the two neighboring stability states, $(\alpha_0,\beta_0,\omega_0)$ and (α,β,ω). We let

$$\omega=\omega_0+\delta\omega, \qquad \alpha=\alpha_0+\delta\alpha, \qquad \beta=\beta_0+\delta\beta \tag{10}$$

in Eq. (6) expand the result, keep linear terms in the perturbation quantities, and obtain

$$\delta\omega=\omega_\alpha\delta\alpha+\omega_\beta\delta\beta \tag{11}$$

where ω_α and ω_β are the components of the complex group velocity in the x and z directions, respectively.

To convert from a spatial to a temporal stability, we take $\delta\alpha=-i\alpha_i$ and $\delta\beta=-i\beta_i$, where α_i and β_i are the imaginary parts of α_0 and β_0 so that α and β are real. Then Eq. (11) can be rewritten as

$$\delta\omega=-i\omega_\alpha\alpha_i-i\omega_\beta\beta_i \tag{12}$$

in agreement with the transformation obtained by Nayfeh and Padhye.[2] It is convenient to express ω_α and ω_β in the form

$$\omega_\alpha=c\cos\phi, \qquad \omega_\beta=c\sin\phi \tag{13}$$

where c and ϕ are complex. Using Eqs. (8) and (13), we write Eq. (12) as

$$\delta\omega=-ic\sigma\cos(\phi-\bar\psi) \tag{14}$$

The imaginary part $\delta\omega_i$ of $\delta\omega$ gives the temporal growth rate, while the real part $\delta\omega_r$ of $\delta\omega$ gives a correction to the frequency ω. Thus, the spatial stability problem corresponding to the complex wavenumbers α_0 and β_0 and the real frequency ω is transformed into a temporal stability problem corresponding to the real wavenumbers $\alpha=\alpha_0-i\alpha_i$ and $\beta=\beta-i\beta_i$ and the complex frequency $\omega+\delta\omega$.

To transform a temporal stability problem corresponding to the real wavenumbers α_0 and β_0 and the complex frequency $\omega_0=\omega_r+\omega_i$ to a spatial stability problem, we put $\delta\omega=-i\omega_i$ in Eq. (11) so that $\omega=\omega_0-i\omega_i=\omega_r$ is real. The result is:

$$-i\omega_i=\omega_\alpha\delta\alpha+\omega_\beta\delta\beta \tag{15}$$

Equation (15) provides two real relations among the four unknowns $\delta\alpha_r$, $\delta\alpha_i$, $\delta\beta_r$, and $\delta\beta_i$. Thus, to determine $\delta\alpha$ and $\delta\beta$, we need two additional real constraints. Let us take these constraints in the form

$$\delta\beta=\delta\alpha\tan\bar\psi \tag{16}$$

where $\bar\psi$ is real. Then, $\bar\psi$ is the growth direction and we can write

$$\delta\alpha=\delta k\cos\bar\psi, \qquad \delta\beta=\delta k\sin\bar\psi \tag{17}$$

where δk is complex. Using Eqs. (13) and (17), we rewrite Eq. (15) as

$$\delta k=-i\omega_i/c\cos(\bar\psi-\phi) \tag{18}$$

Hence, the growth rate σ in the $\bar\psi$ direction is given by $\sigma=-$ Imag. δk or

$$\sigma=\omega_i\mathrm{Real}[c\cos(\bar\psi-\phi)]^{-1} \tag{19}$$

in agreement with that derived by Nayfeh and Padhye.[2] Thus, the temporal stability problem is transformed into a spatial stability problem with the growth rate σ as given by Eq. (19) in the $\bar\psi$ direction corresponding to the frequency ω_r and the wavenumbers

$$\alpha=\alpha_0+\delta k_r\cos\bar\psi, \qquad \beta=\beta_0+\delta k_r\sin\bar\psi \tag{20}$$

We note that the wave orientation has been changed from $\psi_1=\tan^{-1}(\beta_0/\alpha_0)$ to $\psi_2=\tan^{-1}(\beta/\alpha)$.

To transform the growth rate from the $\bar\psi_1$ to the $\bar\psi_2$ direction, we apply Eq. (18) for these directions. Thus,

$$\delta k_1=-i\omega_i/c\cos(\bar\psi_1-\phi), \quad \delta k_2=-i\omega_i/c\cos(\bar\psi_2-\phi) \tag{21}$$

Hence,

$$\delta k_2/\delta k_1=\cos(\bar\psi_1-\phi)/\cos(\bar\psi_2-\phi)$$

If we let $\delta k_1=-i\sigma_{s1}$, then

$$\delta k_2=-i\sigma_{s_1}\cos(\bar\psi_1-\phi)/\cos(\bar\psi_2-\phi)$$

Putting $\delta k_2=\delta k_{2r}-i\sigma_{s2}$, we find that

$$\sigma_{s_2}=\sigma_{s_1}Re[\cos(\bar\psi_1-\phi)/\cos(\bar\psi_2-\phi)] \tag{22}$$

in agreement with Eq. (66) of Ref. 2.

These transformations were verified numerically by Nayfeh and Padhye[2] for two- and three-dimensional incompressible mean flows and by El-Hady and Nayfeh[10] for two-dimensional subsonic and supersonic mean flows.

F. Three-Dimensional Compressible Flows

Mack[29] and Lekoudis[30] presented numerical calculations evaluating the effects of compressibility on the stability of the boundary-layer flow over an infinite sweptback wing. Their results show that, in the forward crossflow instability region, the maximum growth rates calculated by using the incompressible stability theory are about 10% higher than those calculated by using the compressible stability theory. In the rear crossflow instability region, the difference between the maximum growth rates calculated by using the incompressible and compressible stability theories is much larger than 14-40%; hence, the use of the incompressible theory is not recommended.[29] In regions of weaker crossflow instability (away from leading and trailing edges), where the instability is dominated by a Tollmien-Schlichting instability, compressibility reduces the maximum amplification rate and considerably changes the wave orientation of the most unstable wave.

II. Problem Formulation

In this paper we develop a theory that can determine the growth direction of three-dimensional waves in three-dimensional, compressible, growing boundary layers. This direction is needed for calculating the wave amplitude and hence correlating transition locations. Lengths, velocities, and time are made dimensionless using a suitable reference length L^*, the freestream velocity U_∞^*, and L^*/U_∞^*, respectively. The pressure is made dimensionless using $\rho_\infty^* U_\infty^{*2}$. The temperature, density, specific heats, viscosity, and thermal conductivity are made dimensionless using their corresponding freestream values.

To study the linear stability of a steady three-dimensional, compressible boundary-layer flow (basic flow), we superpose a small time-dependent disturbance on each mean-flow, thermodynamic, and transport quantity. Thus, we let

$$\hat{q}(x,y,z,t) = Q_s(x,y,z) + q(x,y,z,t) \tag{23}$$

where $Q_s(x,y,z)$ is a three-dimensional basic-state quantity and $q(x,y,z,t)$ is a three-dimensional unsteady disturbance quantity. Here, $\hat{q}$ stands for the velocity components ($u,v,$ and w), temperature T, pressure p, density ρ, and viscosity μ. Substituting Eq. (23) into the dimensionless Navier-Stokes and state equations, subtracting the basic-state quantities, and linearizing the resulting equations in the q's, we obtain the following disturbance equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho_s u + \rho U_s) + \frac{\partial}{\partial y}(\rho_s v + \rho V_s) + \frac{\partial}{\partial z}(\rho_s w + \rho W_s) = 0 \tag{24}$$

$$\rho_s\left(\frac{\partial u}{\partial t} + U_s\frac{\partial u}{\partial x} + u\frac{\partial U_s}{\partial x} + V_s\frac{\partial u}{\partial y} + v\frac{\partial U_s}{\partial y} + W_s\frac{\partial u}{\partial z} + w\frac{\partial U_s}{\partial z}\right) + \rho\left(U_s\frac{\partial U_s}{\partial x} + V_s\frac{\partial U_s}{\partial y} + W_s\frac{\partial U_s}{\partial z}\right) = -\frac{\partial p}{\partial x}$$
$$+\frac{1}{R}\left\{\frac{\partial}{\partial x}\left[\mu_s\left(r\frac{\partial u}{\partial x} + m\frac{\partial v}{\partial y} + m\frac{\partial w}{\partial z}\right) + \mu\left(r\frac{\partial U_s}{\partial x} + m\frac{\partial V_s}{\partial y} + m\frac{\partial W_s}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu_s\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \mu\left(\frac{\partial U_s}{\partial y} + \frac{\partial V_s}{\partial x}\right)\right]\right.$$
$$\left.+\frac{\partial}{\partial z}\left[\mu_s\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \mu\left(\frac{\partial W_s}{\partial x} + \frac{\partial U_s}{\partial z}\right)\right]\right\} \tag{25}$$

$$\rho_s\left(\frac{\partial v}{\partial t} + U_s\frac{\partial v}{\partial x} + u\frac{\partial V_s}{\partial x} + V_s\frac{\partial v}{\partial y} + v\frac{\partial V_s}{\partial y} + W_s\frac{\partial v}{\partial z} + w\frac{\partial V_s}{\partial z}\right) + \rho\left(U_s\frac{\partial V_s}{\partial x} + V_s\frac{\partial V_s}{\partial y} + W_s\frac{\partial V_s}{\partial z}\right) = -\frac{\partial p}{\partial y} + \frac{1}{R}\left\{\frac{\partial}{\partial x}\left[\mu_s\left(\frac{\partial u}{\partial y}\right.\right.\right.$$
$$\left.+\frac{\partial v}{\partial x}\right) + \mu\left(\frac{\partial U_s}{\partial y} + \frac{\partial V_s}{\partial x}\right)\Big] + \frac{\partial}{\partial y}\left[\mu_s\left(m\frac{\partial u}{\partial x} + r\frac{\partial v}{\partial y} + m\frac{\partial w}{\partial z}\right)\right.$$
$$\left.+\mu\left(m\frac{\partial U_s}{\partial x} + r\frac{\partial V_s}{\partial y} + m\frac{\partial W_s}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\mu_s\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right.$$
$$\left.\left.+\mu\left(\frac{\partial V_s}{\partial z} + \frac{\partial W_s}{\partial y}\right)\right]\right\} \tag{26}$$

$$\rho_s\left(\frac{\partial w}{\partial t} + U_s\frac{\partial w}{\partial x} + u\frac{\partial W_s}{\partial x} + V_s\frac{\partial w}{\partial y} + v\frac{\partial W_s}{\partial y} + W_s\frac{\partial w}{\partial z} + w\frac{\partial W_s}{\partial z}\right) + \rho\left(U_s\frac{\partial W_s}{\partial x} + V_s\frac{\partial W_s}{\partial y} + W_s\frac{\partial W_s}{\partial z}\right) = -\frac{\partial p}{\partial z} + \frac{1}{R}\left\{\frac{\partial}{\partial x}\left[\mu_s\left(\frac{\partial w}{\partial x}\right.\right.\right.$$
$$\left.+\frac{\partial u}{\partial z}\right) + \mu\left(\frac{\partial W_s}{\partial x} + \frac{\partial U_s}{\partial z}\right)\Big] + \frac{\partial}{\partial y}\left[\mu_s\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right.$$
$$\left.+\mu\left(\frac{\partial V_s}{\partial z} + \frac{\partial W_s}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu_s\left(m\frac{\partial u}{\partial x} + m\frac{\partial v}{\partial y} + r\frac{\partial w}{\partial z}\right)\right]$$
$$\left.+\frac{\partial}{\partial z}\left[\mu\left(m\frac{\partial U_s}{\partial x} + m\frac{\partial V_s}{\partial y} + r\frac{\partial W_s}{\partial z}\right)\right]\right\} \tag{27}$$

$$\rho_s\left[\frac{\partial T}{\partial t} + u\frac{\partial T_s}{\partial x} + U_s\frac{\partial T}{\partial x} + v\frac{\partial T_s}{\partial y} + V_s\frac{\partial T}{\partial y} + w\frac{\partial T_s}{\partial z} + W_s\frac{\partial T}{\partial z}\right]$$
$$+\rho\left[U_s\frac{\partial T_s}{\partial x} + V_s\frac{\partial T_s}{\partial y} + W_s\frac{\partial T_s}{\partial z}\right] = (\gamma - 1)M_\infty^2\left[\frac{\partial p}{\partial t} + u\frac{\partial p_s}{\partial x}\right.$$
$$\left.+U_s\frac{\partial p}{\partial x} + V_s\frac{\partial p}{\partial y} + w\frac{\partial p_s}{\partial z} + W_s\frac{\partial p}{\partial z} + \frac{1}{R}\Phi\right] + \frac{1}{RPr}$$
$$\times\left[\frac{\partial}{\partial x}\left(\mu_s\frac{\partial T}{\partial x} + \mu\frac{\partial T_s}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu_s\frac{\partial T}{\partial y} + \mu\frac{\partial T_s}{\partial y}\right)\right.$$
$$\left.+\frac{\partial}{\partial z}\left(\mu_s\frac{\partial T}{\partial z} + \mu\frac{\partial T_s}{\partial z}\right)\right] \tag{28}$$

$$\frac{p}{p_s} = \frac{T}{T_s} + \frac{\rho}{\rho_s} \tag{29}$$

where Φ, the perturbation dissipation function, is defined as

$$\Phi = \mu_s\left\{2r\left(\frac{\partial U_s}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial V_s}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial W_s}{\partial z}\frac{\partial w}{\partial z}\right)\right.$$
$$+2m\left[\frac{\partial U_s}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{\partial V_s}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \frac{\partial W_s}{\partial z}\left(\frac{\partial u}{\partial x}\right.\right.$$
$$\left.\left.+\frac{\partial v}{\partial y}\right)\right] + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\frac{\partial U_s}{\partial y} + \frac{\partial V_s}{\partial x}\right)$$
$$+2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\left(\frac{\partial U_s}{\partial z} + \frac{\partial W_s}{\partial x}\right)$$
$$\left.+2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\left(\frac{\partial V_s}{\partial z} + \frac{\partial W_s}{\partial y}\right)\right\} + \mu\left\{r\left[\left(\frac{\partial U_s}{\partial x}\right)^2 + \left(\frac{\partial V_s}{\partial y}\right)^2\right.\right.$$
$$\left.+\left(\frac{\partial W_s}{\partial z}\right)^2\right] + 2m\left[\frac{\partial U_s}{\partial x}\frac{\partial V_s}{\partial y} + \frac{\partial U_s}{\partial x}\frac{\partial V_s}{\partial y} + \frac{\partial V_s}{\partial y}\frac{\partial W_s}{\partial z}\right]$$
$$\left.+\left(\frac{\partial U_s}{\partial y} + \frac{\partial V_s}{\partial x}\right)^2 + \left(\frac{\partial U_s}{\partial z} + \frac{\partial W_s}{\partial x}\right)^2 + \left(\frac{\partial V_s}{\partial z} + \frac{\partial W_s}{\partial y}\right)^2\right\} \tag{30}$$

Moreover, r and m are given by

$$r = \tfrac{2}{3}(e+2) \qquad m = \tfrac{2}{3}(e-1)$$

where $e = 0$ corresponds to the Stokes hypothesis.

The problem is completed by the specification of the boundary and initial conditions. The boundary conditions are

$$u = v = w = T = 0 \quad \text{at} \quad y = 0 \tag{31}$$

$$u,v,w,T \to 0 \quad \text{as} \quad y \to \infty \tag{32}$$

where the disturbances are assumed to be subsonic. For the initial conditions, we consider two cases. First, we consider the disturbance to be generated by a source (such as a vibrating ribbon) oscillating at the frequency ω (i.e., monochromatic wave) at the curve defined by $x = 0$. Thus, we can express this initial condition for the pressure as

$$p(x,y,z,t) = \tilde{P}(y,z)\exp(-i\omega t) \quad \text{at} \quad x = 0 \tag{33}$$

where ω is real. It is convenient to express the z variation in Eq. (33) as a Fourier integral; that is,

$$p(x,y,z,t) = \int \hat{P}(y,\beta)\exp[i(\beta z - \omega t)\mathrm{d}\beta \text{ at } x=0 \tag{34}$$

where β takes on all possible complex values. Second, we consider a general initial disturbance (wavepacket) at the curve defined by $x=0$. Thus, it can be expressed by the double Fourier integral

$$p(x,y,z,t) = \iint \hat{P}(y,\beta,\omega)\exp[i(\beta z - \omega t)]\mathrm{d}\beta\mathrm{d}\omega \text{ at } x=0 \tag{35}$$

We restrict our analysis to mean flows that are slightly nonparallel; that is, the normal velocity component is small compared with the streamwise velocity component. This condition demands all mean-flow variables to be weak functions of the streamwise and spanwise positions. These assumptions are expressed mathematically by writing the mean-flow variables in the form

$$U_s = U_s(x_1,z_1,y), \quad V_s = \epsilon V_s^*(x_1,z_1,y), W_s = W_s(x_1,y,z_1),$$

$$p_s = p_s(x_1,z_1), \quad T_s = T_s(x_1,z_1,y) \tag{36}$$

where the slow scales $x_1 = \epsilon x$ and $z_1 = \epsilon x$ are used to describe the relatively slow variations of the mean-flow quantities. In what follows, the asterisk will be dropped for simplicity of notation. The viscosity coefficient is assumed to be a function of the temperature only, so that we can write

$$\mu = T\frac{\mathrm{d}\mu_s}{\mathrm{d}T}(T_s) = \mu_s' T \tag{37}$$

III. Mean Flows Independent of Spanwise Direction

In this case, Eqs. (24-32) are independent of z and t. Since these equations, as well as the initial conditions, Eqs. (34) and (35), are linear, the initial conditions suggest a separable solution in the form

$$q = \bar{q}(x,x_1,y)\exp[i(\beta x - \omega t)] \tag{38}$$

where β and ω are constants. Then, using the method of multiple scales,[28] one can determine an approximate solution to these equations. For example, the pressure can be expressed as

$$\bar{p} = \zeta(y;x_1)\exp[i\int\alpha(x_1)\mathrm{d}x] \tag{39}$$

where ζ is a known eigenfunction and α is a known functional of the eigenfunctions. Hence,

$$\hat{p} = \zeta(y;x_1)\exp[i\int\alpha(x_1)\mathrm{d}x + i\beta z - i\omega t] \tag{40}$$

Equation (40) shows that the wave orientation angle $\psi = \tan^{-1}(\beta/\alpha)$ is not a constant but a varying function of the streamwise location. Consequently, one should not fix the wave orientation but fix β for such flows. For the case of two-dimensional compressible flows, Mack[26] found that for a given frequency the wave orientation can change by a few degrees from neutral point to neutral point. This change has a substantial effect on the maximum amplitude ratio determined from parallel[26] and nonparallel[10] stability theories.

Near the leading edge of a sweptback wing, the mean flow and, hence, α vary appreciably with the streamwise location. However, away from the leading edge, α_r/R is approximately constant. Hence, α_r varies slowly with the streamwise position. Consequently, the wavelength of the disturbance in the midchord region is approximately constant.

IV. Integral Representation and Saddle-Point Method

When the mean flow is parallel, α and ζ are independent of x_1, and Eq. (40) becomes

$$\hat{p} = \zeta(y;\beta)\exp[i(\alpha x + \beta z - \omega t)] \tag{41}$$

Summing over all possible values of β, we find that for a monochromatic disturbance

$$p(x,y,z,t) = \exp(-i\omega t)\int\zeta(y;\beta)\exp[i\alpha(\beta)x + i\beta z]\mathrm{d}\beta \tag{42}$$

where $\alpha = \alpha(\beta)$. To evaluate the integral in Eq. (42), one needs to solve the eigenvalue problem for all possible values of β and then perform the integration—an expensive procedure. However, far from the disturbance (i.e., large x), we can use an asymptotic analysis (namely, the saddle-point method) to determine an approximation to the integral. To this end, we write the exponent in the integral in Eq. (42) as

$$i\alpha(\beta)x + i\beta z = ix\chi \tag{43}$$

where

$$\chi = \alpha(\beta) + z\beta/x \tag{44}$$

As $x \to \infty$ with z/x being constant, the leading term in the asymptotic expansion of the integral in Eq. (42) arises from the immediate neighborhoods of the saddle points.[31] They are given by $\partial\chi/\partial\beta = 0$, or

$$\mathrm{d}\alpha/\mathrm{d}\beta + z/x = 0 \tag{45}$$

When Eq. (45) is satisfied by one value of β denoted by β^*, the asymptotic representation of Eq. (42) yields

$$p(x,y,z,t) \propto \frac{\exp[i\alpha(\beta^*)x + i\beta^* z - i\omega t]}{[x(\partial^2\alpha/\partial\beta^2)(\beta^*)]^{1/2}} \tag{46}$$

The solution given by Eqs. (45) and (46) is general and applicable to complex values of x and z. For the problem under consideration, x and z are real; hence, only the solution of Eq. (45) for real x and z provides a representation in the physical space. Thus, for the physical problem, the imaginary part of $\mathrm{d}\alpha/\mathrm{d}\beta$ must be zero. Consequently,

$$\mathrm{d}\alpha/\mathrm{d}\beta = -z/x = \text{real number} \tag{47}$$

When α is an analytic function of β, we can use the Cauchy-Riemann conditions to rewrite Eq. (47) as

$$\partial\alpha_r/\partial\beta_r = -z/x \tag{48a}$$

$$\partial\alpha_i/\partial\beta_r = 0 \tag{48b}$$

Equations (48) define the complex quantity $\beta^* = \beta_r^* + i\beta_i^*$ in terms of z/x. Moreover, along a given ray in the xz plane, β^* is constant. Thus, for example, to calculate the amplitude ratio and hence the factor n for transition correlations, one divides the β_r^* axes into intervals and identifies these intervals by their midpoints. Moreover, with each interval is associated a group of waves. Using the midpoint value, one iterates on the value of β_i^* that satisfies Eq. (48b). This fixes the value of β_i^* and, hence, determines z/x and the corresponding ray in the xz plane. Then, it follows from Eq. (46), that the amplitude a of the wave along this ray is

$$a \propto \frac{\exp[-\alpha_i(\beta^*)x - \beta_i^* z]}{[x(\partial^2\alpha/\partial\beta^2)(\beta^*)]^{1/2}}$$

$$= \frac{\exp\{-[\alpha_i(\beta^*) - \beta_i^*(\partial\alpha_r/\partial\beta_r)(\beta^*)]x\}}{[x(\partial^2\alpha/\partial\beta^2)(\beta^*)]^{1/2}} \tag{49}$$

It should be mentioned that the preceding saddle-point analysis is valid only for parallel (i.e., nongrowing) boundary layers. For a growing boundary layer, we obtain next an asymptotic solution by using the method of multiple scales.

V. Method of Multiple Scales

Instead of outlining the analysis for a monochromatic wave, we outline it for a wavepacket so that we do not need to repeat it again. Thus, we seek a solution to Eqs. (24-32), (36), and (37) in the form

$$q(x,y,z,t) = [q_0(x_1,z_1,t_1,y) + \epsilon q_1(x_1,z_1,t_1,y) + \ldots]\exp(i\theta) \tag{50}$$

where $t_1 = \epsilon t$ and

$$\frac{\partial \theta}{\partial t} = -\omega, \quad \frac{\partial \theta}{\partial x} = \alpha(x_1,z_1), \quad \frac{\partial \theta}{\partial z} = \beta(x_1,z_1) \tag{51}$$

Assuming that θ is continuously differentiable, we have

$$\frac{\partial \alpha}{\partial z_1} = \frac{\partial \beta}{\partial x_1} \tag{52}$$

Substituting Eqs. (50) and (51) into Eqs. (24-32), using Eqs. (36) and (37), and equating the coefficients of ϵ^0 and ϵ on both sides, we obtain equations describing the zeroth- and first-order problems. Using the notation

$$z_{01} = u_0, \quad z_{02} = Du_0, \quad z_{03} = v_0, \quad z_{04} = p_0$$
$$z_{05} = T_0, \quad z_{06} = DT_0, \quad z_{07} = w_0, \quad z_{08} = Dw_0 \tag{53}$$

we write the zeroth-order problem in the compact form

$$Dz_{0n} - \sum_{m=1}^{8} a_{nm} z_{0_m} = 0 \quad \text{for} \quad n = 1,2,\ldots,8 \tag{54}$$

$$z_{01} = z_{03} = z_{07} = 0 \quad \text{and} \quad z_{05} = 0 \quad \text{at} \quad y = 0 \tag{55}$$

$$z_{01}, z_{03}, z_{05}, z_{07}, \to 0 \quad \text{as} \quad y \to \infty \tag{56}$$

where $D = \partial/\partial y$ and the a_{nm} are the elements of an 8×8 variable-coefficient matrix whose nonzero elements are given in Appendix A. Given R and two real relations among the parameters $\alpha_r, \alpha_i, \beta_r$, and β_i, the eigenvalue problem, Eqs. (54-56), is usually solved numerically to provide two other real relations among these parameters and an eigenvector solution that can be expressed in the form

$$z_{0m} = A(x_1,z_1,t_1)\zeta_m(x_1,y,z_1) \tag{57}$$

The amplitude function $A(x_1,z_1,t_1)$ is arbitrary at this order; it is determined by imposing the solvability condition at the next order.

Using Eq. (57), we write the first-order problem as

$$Dz_{1n} - \sum_{m=1}^{8} a_{nm} z_{1_m} = D_n \frac{\partial A}{\partial t_1} + E_n \frac{\partial A}{\partial x_1} + F_n \frac{\partial A}{\partial z_1} + G_n A$$
$$\text{for} \quad n = 1,2,\ldots,8 \tag{58}$$

$$z_{11} = z_{13} = z_{17} = 0, \quad z_{15} = 0 \quad \text{at} \quad y = 0 \tag{59}$$

$$z_{11}, z_{13}, z_{15}, z_{17} \to 0 \quad \text{as} \quad y \to \infty \tag{60}$$

where the D_n, E_n, F_n, and G_n are known functions of the ζ_n, α, β, and the mean-flow quantities; they are given in Appendix B. Since the homogeneous parts of Eqs. (58-60) are the same as Eqs. (54-56) and since the latter have a nontrivial solution, the inhomogeneous equations (58-60) have a solution if and only if the inhomogeneous parts are orthogonal to every solution of the adjoint homogeneous problem; that is

$$\int_0^\infty \sum_{n=1}^{\infty} \left[D_n \frac{\partial A}{\partial t_1} + E_n \frac{\partial A}{\partial x_1} + F_n \frac{\partial A}{\partial z_1} + G_n A\right]\zeta_n^* \mathrm{d}y = 0 \tag{61}$$

where the $\zeta_n^*(x_1,y,z_1)$ are solutions of the adjoint homogeneous problem

$$D\zeta_n^* + \sum_{m=1}^{8} a_{mn}\zeta_m^* = 0 \quad \text{for} \quad n = 1,2,\ldots,8 \tag{62}$$

$$\zeta_2^* = \zeta_4^* = \zeta_8^* = 0, \quad \zeta_6^* = 0 \quad \text{at} \quad y = 0 \tag{63}$$

$$\zeta_2^*, \zeta_4^*, \zeta_6^*, \zeta_8^* \to 0 \quad \text{at} \quad y \to \infty \tag{64}$$

Substituting for the D_n, E_n, F_n, and G_n from Appendix B into Eq. (61), and rewriting the result in terms of the original variables, we obtain

$$\frac{\partial A}{\partial t} + \omega_\alpha \frac{\partial A}{\partial x} + \omega_\beta \frac{\partial A}{\partial z} = \epsilon h_1(\alpha,\beta,x,z)A \tag{65}$$

where ω_α and ω_β are the components of the complex group velocity in the x and z directions; they, along with h_1, are given in quadratures, as in Appendix C.

To determine the equations describing α and β, we replace z_{0n} with ζ_n in Eqs. (54-56), differentiate the result with respect to x_1, and obtain

$$D\left(\frac{\partial \zeta_n}{\partial x_1}\right) - \sum_{m=1}^{8} a_{nm}\left(\frac{\partial \zeta_m}{\partial x_1}\right) = iE_n \frac{\partial \alpha}{\partial x_1} + iF_n \frac{\partial \beta}{\partial x_1} + \sum_{m=1}^{8} \frac{\partial a_{nm}}{\partial x_1}\bigg|_{\alpha,\beta} \zeta_m \tag{66}$$

$$\frac{\partial \zeta_1}{\partial x_1} = \frac{\partial \zeta_3}{\partial x_1} = \frac{\partial \zeta_7}{\partial x_1} = 0, \quad \frac{\partial \zeta_5}{\partial x_1} = 0 \quad \text{at} \quad y = 0 \tag{67}$$

$$\frac{\partial \zeta_1}{\partial x_1}, \frac{\partial \zeta_3}{\partial x_1}, \frac{\partial \zeta_5}{\partial x_1}, \frac{\partial \zeta_7}{\partial x_1} \to 0 \quad \text{as} \quad y \to \infty \tag{68}$$

Applying the solvability condition to Eqs. (66-68), we obtain

$$\omega_\alpha \frac{\partial \alpha}{\partial x_1} + \omega_\beta \frac{\partial \beta}{\partial x_1} = h_2(\alpha,\beta,x_1,z_1) \tag{69}$$

where h_2 is given in Appendix C. Similarly, differentiating Eqs. (54-56) with respect to z_1 knd applying the solvability condition to the resulting problem, we obtain

$$\omega_\alpha \frac{\partial \alpha}{\partial z_1} + \omega_\beta \frac{\partial \beta}{\partial z_1} = h_3(\alpha,\beta,x_1,z_1) \tag{70}$$

where h_3 is given in Appendix C. Using Eq. (52), we rewrite Eqs. (69) and (70) in terms of x and z as

$$\omega_\alpha \frac{\partial \alpha}{\partial x} + \omega_\beta \frac{\partial \alpha}{\partial z} = \epsilon h_2(\alpha,\beta,x,z) \tag{71}$$

$$\omega_\alpha \frac{\partial \beta}{\partial x} + \omega_\beta \frac{\partial \beta}{\partial z} = \epsilon h_3(\alpha,\beta,x,z) \tag{72}$$

Equations (65, 71, and 72) describe the evolution of the amplitude, phase, and wavenumbers of the disturbance.

For a monochromatic wave, ω is constant and Eq. (65) reduces to

$$\omega_\alpha \frac{\partial A}{\partial x} + \omega_\beta \frac{\partial A}{\partial z} = \epsilon h_1 A \tag{73}$$

In general, $\omega_\beta/\omega_\alpha$ is complex and Eqs. (71-73) are elliptic for real x and z. When x and z are analytically continued in the complex plane, Eqs. (71-73) can be rewritten as

$$\frac{\mathrm{d}A}{\mathrm{d}s} = \epsilon h_1(\alpha,\beta,x,z)A \tag{74a}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = \epsilon h_2(\alpha,\beta,x,z) \tag{74b}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}s} = \epsilon h_3(\alpha,\beta,x,z) \tag{74c}$$

where

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \omega_\alpha, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = \omega_\beta \tag{75}$$

It follows from Eq. (75) that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\omega_\beta}{\omega_\alpha} \tag{76}$$

Hence, Eqs. (74) and (75) represent a physical problem only when $\omega_\beta/\omega_\alpha$ is real because z and x are real.

For a parallel flow, condition (76) reduces to condition (47) obtained by the saddle-point method. To see this, we differentiate Eq. (6) with respect to β, note that ω is constant, and obtain

$$\omega_\alpha \frac{\mathrm{d}\alpha}{\mathrm{d}\beta} + \omega_\beta = 0 \tag{77a}$$

Hence,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\beta} = -\frac{\omega_\beta}{\omega_\alpha} \tag{77b}$$

and Eq. (76) can be rewritten as

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{\mathrm{d}\alpha}{\mathrm{d}\beta} = \text{a real quantity} \tag{77c}$$

which defines the propagation of the wave.

For growing boundary layers, β and α vary along the rays as in Eqs. (74). The equations describing α and β are nonlinear coupled differential equations that need to be solved simultaneously. Usually, α and β are determined by solving the eigenvalue problem rather than these differential equations. For a first-order estimate, one may approximate α and β by

$$\alpha = \epsilon \int h_2 \mathrm{d}s \qquad \beta = \epsilon \int h_3 \mathrm{d}s \tag{78a}$$

Moreover, it follows from Eq. (74a) that

$$A = A_0 \exp(\epsilon \int h_1 \mathrm{d}s) \tag{78b}$$

where A_0 is a constant. Using Eqs. (51, 57, 76, and 78b) in Eq. (50), we obtain

$$u = A_0 \zeta_1(x_1,z_1;y) \exp\left[i\int\left(\alpha + \beta\frac{\omega_\beta}{\omega_\alpha}\right)\mathrm{d}x + \epsilon\int \frac{h_1}{\omega_\alpha}\mathrm{d}x - i\omega t\right] \tag{79}$$

The amplification of the wave, as in the two-dimensional case, depends on the normal distance from the wall. Moreover, it depends on the spanwise location. In the quasiparallel approximation, ζ_1 is independent of x_1 and z_1, and $h_1 = 0$.

VI. Wavepackets

For a wavepacket, the disturbance consists of many frequencies and it can be represented for the case of parallel flows by the following integral for the case of spatial stability:

$$p(x,y,z,t) = \iint \zeta(y;\beta,\omega) \exp[i\alpha(\beta,\omega)x + i\beta z - i\omega t]\mathrm{d}\omega \mathrm{d}\beta \tag{80}$$

where the integration extends over all complex values of ω and β. Using the saddle-point method,[31] we can write an asymptotic expansion for Eq. (80) valid for large x with z/x and x/t being constants as

$$p(x,y,z,t) \propto \frac{\exp[i\alpha(\beta^*,\omega^*)x + i\beta^* z - i\omega^* t]}{x\left[\dfrac{\partial^2\alpha}{\partial\beta^2}\dfrac{\partial^2\alpha}{\partial\omega^2} - \left(\dfrac{\partial^2\alpha}{\partial\beta\partial\omega}\right)^2\right]^{1/2}} \tag{81}$$

The saddle point (β^*,ω^*) is defined by setting the partial derivatives of the exponent $\alpha x + \beta z - \omega t$ with respect to ω and β equal to zero. The result is

$$\frac{z}{x} = -\frac{\partial\alpha}{\partial\beta}, \qquad \frac{t}{x} = \frac{\partial\alpha}{\partial\omega} \tag{82}$$

In order that Eqs. (81) and (82) represent the physical problem, $\partial\alpha/\partial\beta$ and $\partial\alpha/\partial\omega$ must be real.

For the case of temporal stability, the integral representation of p can be written as

$$p(x,y,z,t) = \iint \zeta(y,\alpha,\beta) \exp[i\alpha x + i\beta z - i\omega(\alpha,\beta)t]\mathrm{d}\alpha \mathrm{d}\beta \tag{83}$$

Again, using the saddle-point method,[31] we can write an asymptotic expansion for Eq. (83) in the form

$$p \propto \frac{\exp[i\alpha^* x + i\beta^* z - i\omega(\alpha^*,\beta^*)t]}{t\left[\dfrac{\partial^2\omega}{\partial\alpha^2}\dfrac{\partial^2\omega}{\partial\beta^2} - \left(\dfrac{\partial^2\omega}{\partial\alpha\partial\beta}\right)^2\right]^{1/2}} \tag{84}$$

where the saddle point is given by

$$\frac{x}{t} = \frac{\partial\omega}{\partial\alpha}, \qquad \frac{z}{t} = \frac{\partial\omega}{\partial\beta} \tag{85}$$

In order that Eqs. (84) and (85) represent the physical problem $\partial\omega/\partial\alpha$ and $\partial\omega/\partial\beta$, the group velocities must be real.

It follows from Eqs. (82) and (85) that the rays passing through a saddle point are defined by complex values of α, β, and ω, except those corresponding to maximum amplification on which $\partial\omega_i/\partial\alpha$ and $\partial\omega_i/\partial\beta$ vanish. Since the dispersion relation is usually calculated from a temporal stability in which α and β are real but ω may be complex, expansion procedures are usually used to define the locations of the saddle points from these calculations for slightly nonconservative systems. For the two-dimensional case, the result is[32]

$$p \propto \frac{\exp[i\alpha_r^* x - i\omega_r(\alpha_r^*)t + \omega_i(\alpha_r^*)t]}{[t\partial^2\omega_r/\partial\alpha_r^2]^{1/2}} \tag{86}$$

where α_r^* is defined by

$$\frac{x}{t} = \frac{\partial\omega_r}{\partial\alpha}(\alpha) \tag{87}$$

For the three-dimensional case, the result is[33,34]

$$p \propto \frac{\exp[i\alpha_r^* x + i\beta_r^* z - i\omega_r(\alpha_r^*,\beta_r^*)t + \omega_i(\alpha_r^*,\beta_r^*)t]}{t\left[\frac{\partial^2\omega_r}{\partial\alpha_r^2}\frac{\partial^2\omega_r}{\partial\beta_r^2} - \left(\frac{\partial^2\omega_r}{\partial\alpha_r\partial\beta_r}\right)^2\right]^{1/2}} \tag{88}$$

where α_r^* and β_r^* are defined by

$$\frac{x}{t} = \frac{\partial\omega_r}{\partial\alpha}(\alpha,\beta), \quad \frac{z}{t} = \frac{\partial\omega_r}{\partial\beta}(\alpha,\beta) \tag{89}$$

Equations (86-89) state that the various components of the spectrum of a disturbance propagate along rays defined by the real parts of the group velocities, as in the conservative case; but in this case, each component amplifies or damps by the amplification rate ω_i appropriate to each ray. To determine the limitations of Eqs. (86-89), we present next the details of their derivations.

For the two-dimensional case, the saddle points are defined by the first of Eq. (85). Let α_0 denote the real part α_r of α and let $\alpha-\alpha_0$ be a small quantity. Then, it follows from the dispersion relation $\omega=\omega(\alpha)$ that

$$\omega = \omega(\alpha_0) + \omega_\alpha(\alpha_0)(\alpha-\alpha_0) + \tfrac{1}{2}\omega_{\alpha\alpha}(\alpha_0)(\alpha-\alpha_0)^2 + \ldots \tag{90}$$

and from Eq. (85) that

$$x/t = \omega_\alpha(\alpha_0) + \omega_{\alpha\alpha}(\alpha_0)(\alpha-\alpha_0) + \ldots \tag{91}$$

Hence,

$$\alpha x - \omega t = \alpha_0 x + (\alpha-\alpha_0)x - \omega(\alpha_0)t - \omega_\alpha(\alpha_0)(\alpha-\alpha_0)t - \tfrac{1}{2}\omega_{\alpha\alpha}(\alpha_0)(\alpha-\alpha_0)^2 t + \ldots \tag{92}$$

Eliminating $\omega_\alpha(\alpha_0)$ from Eqs. (91) and (92) yields

$$\alpha x - \omega t = \alpha_0 x - \omega(\alpha_0)t + \tfrac{1}{2}\omega_{\alpha\alpha}(\alpha_0)(\alpha-\alpha_0)^2 t + \ldots \tag{93}$$

Putting $\omega=\omega_r+i\omega_i$, where $\omega_i \ll \omega_r$, in Eq. (91), we have

$$\frac{x}{t} = \frac{\partial\omega_r}{\partial\alpha}(\alpha_0) + i\frac{\partial\omega_i}{\partial\alpha}(\alpha_0) + \frac{\partial^2\omega_r}{\partial\alpha^2}(\alpha_0)(\alpha-\alpha_0) + i\frac{\partial^2\omega_i}{\partial\alpha^2}(\alpha_0)(\alpha-\alpha_0) + \ldots \tag{94}$$

When the derivatives of ω_i are small compared with the derivatives of ω_r, one can determine the following approximate expression:

$$\alpha-\alpha_0 = -i\frac{\partial\omega_i}{\partial\alpha}(\alpha_0)\Big/\frac{\partial^2\omega_r}{\partial\alpha^2}(\alpha_0) \tag{95}$$

where

$$\frac{x}{t} = \frac{\partial\omega_r}{\partial\alpha}(\alpha_0) \tag{96}$$

Thus, Eq. (95) is a uniform expansion only when the above conditions are satisfied. They are satisfied for rays near that corresponding to the maximum amplification, but may not be satisfied for the other rays. Substituting for $(\alpha-\alpha_0)$ from Eq. (95) into Eq. (93) yields

$$\alpha x - \omega t = \alpha_0 x - \omega(\alpha_0)t + \frac{[(\partial\omega_i/\partial\alpha)(\alpha_0)]^2 t}{2(\partial^2\omega_r/\partial\alpha^2)(\alpha_0)} + \ldots \tag{97}$$

Comparing Eq. (97) with the argument of the exponent in Eq. (86) shows that there is an extra term in Eq. (97) which is negligible only when $(\partial\omega_i/\partial\alpha)^2 \ll \partial^2\omega_r/\partial\alpha^2$; this condition is less restrictive than the condition necessary for the uniformity of Eq. (95).

For the three-dimensional case, it follows from the dispersion relation $\omega=\omega(\alpha,\beta)$ that

$$\omega = \omega(0) + \omega_\alpha(0)(\alpha-\alpha_0) + \omega_\beta(0)(\beta-\beta_0) + \tfrac{1}{2}\omega_{\alpha\alpha}(0)(\alpha-\alpha_0)^2 + \omega_{\alpha\beta}(0)(\alpha-\alpha_0)(\beta-\beta_0) + \tfrac{1}{2}\omega_{\beta\beta}(0)(\beta-\beta_0)^2 + \ldots \tag{98}$$

and from Eqs. (85) that

$$x/t = \omega_\alpha(0) + \omega_{\alpha\alpha}(0)(\alpha-\alpha_0) + \omega_{\alpha\beta}(0)(\beta-\beta_0) + \ldots \tag{99}$$

$$z/t = \omega_\beta(0) + \omega_{\alpha\beta}(0)(\alpha-\alpha_0) + \omega_{\beta\beta}(0)(\beta-\beta_0) + \ldots \tag{100}$$

where $F(0)$ stands for $F(\alpha_0,\beta_0)$. Hence,

$$\begin{aligned}\alpha x + \beta z - \omega t &= \alpha_0 x + \beta_0 z - \omega(0)t + (\alpha-\alpha_0)x + (\beta-\beta_0)z \\ &\quad - \omega_\alpha(0)(\alpha-\alpha_0)t - \omega_\beta(0)(\beta-\beta_0)t - \tfrac{1}{2}\omega_{\alpha\alpha}(0)(\alpha-\alpha_0)^2 t \\ &\quad - \omega_{\alpha\beta}(0)(\alpha-\alpha_0)(\beta-\beta_0)t - \tfrac{1}{2}\omega_{\beta\beta}(0)(\beta-\beta_0)^2 t + \ldots \\ &= \alpha_0 x + \beta_0 z - \omega(\alpha_0,\beta_0)t + \tfrac{1}{2}\omega_{\alpha\alpha}(0)(\alpha-\alpha_0)^2 t \\ &\quad + \omega_{\alpha\beta}(0)(\alpha-\alpha_0)(\beta-\beta_0)t + \tfrac{1}{2}\omega_{\beta\beta}(0)(\beta-\beta_0)^2 t + \ldots\end{aligned} \tag{101}$$

Assuming that $\omega=\omega_r+i\omega_i$, where $\omega_i \ll \omega_r$ and the derivatives of ω_i are small compared with the derivatives of ω_r, we find from Eqs. (99) and (100) that

$$\alpha-\alpha_0 = -i\left[\frac{\partial\omega_i}{\partial\alpha}\frac{\partial^2\omega_r}{\partial\beta^2} - \frac{\partial\omega_i}{\partial\beta}\frac{\partial^2\omega_r}{\partial\alpha\partial\beta}\right]\left[\frac{\partial^2\omega_r}{\partial\alpha^2}\frac{\partial^2\omega_r}{\partial\beta^2} - \left(\frac{\partial^2\omega_r}{\partial\alpha\partial\beta}\right)^2\right]^{-1} \tag{102}$$

$$\beta-\beta_0 = -i\left[\frac{\partial\omega_i}{\partial\beta}\frac{\partial^2\omega_r}{\partial\alpha^2} - \frac{\partial\omega_i}{\partial\alpha}\frac{\partial^2\omega_r}{\partial\alpha\partial\beta}\right]\left[\frac{\partial^2\omega_r}{\partial\alpha^2}\frac{\partial^2\omega_r}{\partial\beta^2} - \left(\frac{\partial^2\omega_r}{\partial\alpha\partial\beta}\right)^2\right]^{-1} \tag{103}$$

where α_0 and β_0 are defined by

$$\frac{x}{t} = \frac{\partial\omega_r}{\partial\alpha}(\alpha,\beta), \quad \frac{z}{t} = \frac{\partial\omega_r}{\partial\beta}(\alpha,\beta) \tag{104}$$

Equations (102) and (103) are uniform expansions if and only if their right-hand sides are small compared with α_0 and β_0. These conditions are satisfied for rays near that corresponding to the maximum amplification, but may not be satisfied for the other rays. Substituting Eqs. (102) and (103) into Eq. (101), we have

$$\begin{aligned}\alpha x + \beta z - \omega t &= \alpha_0 x + \beta_0 z - \omega(\alpha_0,\beta_0)t - \frac{1}{2}\left[\left(\frac{\partial\omega_i}{\partial\alpha}\right)^2\frac{\partial^2\omega_r}{\partial\beta^2} - 2\frac{\partial\omega_i}{\partial\alpha}\frac{\partial\omega_i}{\partial\beta}\frac{\partial^2\omega_r}{\partial\alpha\partial\beta} + \left(\frac{\partial\omega_i}{\partial\beta}\right)^2\frac{\partial^2\omega_r}{\partial\alpha^2}\right] \\ &\quad \times \left[\frac{\partial^2\omega_r}{\partial\alpha^2}\frac{\partial^2\omega_r}{\partial\beta^2} - \left(\frac{\partial^2\omega_r}{\partial\alpha\partial\beta}\right)^2\right]^{-1} t + \ldots\end{aligned} \tag{105}$$

Comparing Eq. (105) with the argument of the exponent in Eq. (88) shows that there is an extra term in Eq. (105) which is negligible only when

$$\left(\frac{\partial\omega_i}{\partial\alpha}\right)^2\frac{\partial^2\omega_r}{\partial\beta^2}-2\frac{\partial\omega_i}{\partial\alpha}\frac{\partial\omega_i}{\partial\beta}\frac{\partial^2\omega_r}{\partial\alpha\partial\beta}$$
$$+\left(\frac{\partial\omega_i}{\partial\beta}\right)^2\frac{\partial^2\omega_r}{\partial\alpha^2}<\frac{\partial^2\omega_r}{\partial\alpha^2}\frac{\partial^2\omega_r}{\partial\beta^2}-\left(\frac{\partial^2\omega_r}{\partial\alpha\partial\beta}\right)^2$$

This condition is satisfied for all rays close to the one corresponding to maximum amplification but may not be satisfied for the other rays.

Gaster[33] used the approximation defined by Eqs. (86) and (87) for the case of unstable waves. He found that caustics develop in the solution when $\partial^2\omega_r(\alpha_0)/\partial\alpha^2$ vanishes. Gaster[35] later pointed out that these caustics are not physical but are the result of the approximations given in Eqs. (86) and (87), because when $\partial\omega_i/\partial\alpha$ is not small compared with $\partial^2\omega_r/\partial\alpha^2$, the expansion, Eqs. (86) and (87), breaks down.

For a growing boundary layer, one needs to use equations such as Eqs. (65, 71, and 72). They can be written in the characteristic form

$$\frac{dA}{dx}=\frac{\epsilon h_1}{\omega_\alpha}A,\quad \frac{d\alpha}{dx}=\frac{\epsilon h_2}{\omega_\alpha},\quad \frac{d\beta}{dx}=\frac{\epsilon h_3}{\omega_\alpha} \tag{106}$$

along the rays

$$\frac{dt}{dx}=\frac{1}{\omega_\alpha}\quad \frac{dz}{dx}=\frac{\omega_\beta}{\omega_\alpha} \tag{107}$$

For the physical problem, ω_α and ω_β must be real. Thus, to calculate the disturbance amplitude and hence n for correlating transition, we divide, for example, the disturbance into groups of waves centered at the real frequency ω_r^* and the real spanwise wavenumber β_r^*. For each group, we iterate on the values of ω_i^* and β_i^* so that the right-hand sides of Eq. (107) are real. This step fixes the values of ω_i^* and β_i^*. The solution of the first of Eqs. (106) along the rays can be expressed as

$$A=A_0\exp\left[\epsilon\int\frac{h_1}{\omega_\alpha}dx\right] \tag{108}$$

Using Eqs. (51, 57, 107, and 108), we rewrite Eq. (50) as

$$u=A_0\zeta\exp\left[i\int\left(\alpha+\frac{\beta\omega_\beta}{\omega_\alpha}-\frac{\omega}{\omega_\alpha}\right)dx+\epsilon\int\frac{h_1}{\omega_\alpha}dx\right] \tag{109}$$

Then, the amplitude of the disturbance can be calculated from Eq. (109). The calculation is repeated for all other groups of waves and the maximum value of the amplitude is used to calculate n and hence correlate the transition location.

Appendix A

$$a_{12}=a_{56}=a_{78}=1,\quad a_{21}=\alpha^2+\beta^2-i\hat{\omega}R/\mu_sT_s,$$
$$a_{22}=-D\mu_s/\mu_s$$
$$a_{23}=-i\alpha(m+1)DT_s/T_s-i\alpha D\mu_s/\mu_s+RDU_s/\mu_sT_s$$
$$a_{24}=i\alpha R/\mu_s+(m+1)\gamma M_\infty^2\alpha\hat{\omega}$$
$$a_{25}=-\alpha(m+1)\hat{\omega}/T_s-D(\mu_s'DU_s)/\mu_s$$
$$a_{26}=-\mu_s'DU_s/\mu_s,\quad a_{31}=-i\alpha,\quad a_{33}=DT_s/T_s,\quad a_{34}=i\gamma M_\infty^2\hat{\omega}$$
$$a_{35}=-i\omega/T_s,\quad a_{37}=-i\beta,\quad a_{41}=-i\chi\alpha(rDT_s/T_s+2D\mu_s/\mu_s)$$
$$a_{42}=-i\chi\alpha,\quad a_{43}=\chi[-\alpha^2-\beta^2+i\hat{\omega}R/\mu_sT_s$$
$$+rD^2T_s/T_s+rD\mu_sDT_s/\mu_sT_s]$$
$$a_{44}=-i\chi r\gamma M_\infty^2[\alpha DU_s+\beta DW_s-\hat{\omega}DT_s/T_s-\hat{\omega}D\mu_s/\mu_s]$$
$$a_{45}=i\chi[r(\alpha DU_s+\beta DW_s)/T_s+\mu_s'(\alpha DU_s+\beta DW_s)/\mu_s$$
$$-r\hat{\omega}D\mu_s/\mu_sT_s]$$
$$a_{46}=-i\chi r\hat{\omega}/T_s,\quad a_{47}=-i\chi r\beta DT_s/T_s-2i\chi\beta D\mu_s/\mu_s$$
$$a_{48}=-i\chi\beta,\quad a_{62}=-2(\gamma-1)M_\infty^2PrDU_s$$
$$a_{63}=-2i(\gamma-1)M_\infty^2Pr(\alpha DU_s+\beta DW_s)+RPrDT_s/\mu_sT_s$$
$$a_{64}=i(\gamma-1)M_\infty^2PrR\hat{\omega}/\mu_s$$
$$a_{65}=\alpha^2+\beta^2-iRPr\hat{\omega}/\mu_sT_s-(\gamma-1)M_\infty^2Pr\mu_s'[(DU_s)^2$$
$$+(DW_s)^2]/\mu_s-D^2\mu_s/\mu_s$$
$$a_{66}=-2D\mu_s/\mu_s,\quad a_{68}=-2(\gamma-1)M_\infty^2PrDW_s$$
$$a_{83}=-i(m+1)\beta DT_s/T_s-i\beta D\mu_s/\mu_s+RDW_s/\mu_sT_s$$
$$a_{84}=(m+1)\gamma M_\infty^2\beta\hat{\omega}+i\beta R/\mu_s$$
$$a_{85}=-(m+1)\beta\hat{\omega}/T_s-D(\mu_s'DW_s)/\mu_s$$
$$a_{86}=-\mu_s'DW_s/\mu_s,\quad a_{87}=\alpha^2+\beta^2-i\hat{\omega}R/\mu_sT_s,$$
$$a_{88}=-D\mu_s/\mu_s$$

where

$$\mu_s'=d\mu_s/dT_s,\quad DF=\partial F/\partial y$$

and

$$\hat{\omega}=\omega-\alpha U_s-\beta W_s,\quad \chi=[R/\mu_s-ir\gamma M_\infty^2\hat{\omega}]^{-1}$$

Appendix B

$$D_n=i\sum_{m-1}^{8}\frac{\partial a_{nm}}{\partial\omega}\zeta_m$$
$$E_n=-i\sum_{m=1}^{8}\frac{\partial a_{nm}}{\partial\alpha}\zeta_m$$
$$F_n=-i\sum_{m=1}^{8}\frac{\partial a_{nm}}{\partial\beta}\zeta_m$$

where small terms $0(R^{-1})$ can be neglected.

$$G_1=G_5=G_7=0$$

$$G_2=\frac{R}{\mu_s}\left[\frac{U_s}{T_s}\frac{\partial\zeta_1}{\partial x_1}+\frac{\zeta_1}{T_s}\frac{\partial U_s}{\partial x_1}+\frac{V_s}{T_s}\frac{\partial\zeta_1}{\partial y_1}+\frac{\zeta_7}{T_s}\frac{\partial U_s}{\partial z_1}\right.$$
$$+\frac{W_s}{T_s}\frac{\partial\zeta_1}{\partial z_1}+\frac{\partial\zeta_4}{\partial x_1}+\left(\frac{\gamma M_\infty^2\zeta_4}{T_s}-\frac{\zeta_5}{T_s^2}\right)\left(U_s\frac{\partial U_s}{\partial x_1}\right.$$
$$\left.\left.+V_s\frac{\partial U_s}{\partial y}+W_s\frac{\partial U_s}{\partial z_1}\right)\right]+0(1)$$

$$G_3=\frac{\zeta_1}{T_s}\frac{\partial T_s}{\partial x_1}-\frac{\partial\zeta_1}{\partial x_1}-U_s\gamma M_\infty^2\left(\frac{\partial\zeta_4}{\partial x_1}-\frac{\zeta_4}{T_s}\frac{\partial T_s}{\partial x_1}\right)+\frac{U_s}{T_s}\frac{\partial\zeta_5}{\partial x_1}$$
$$-\frac{2U_s\zeta_5}{T_s^2}\frac{\partial T_s}{\partial x_1}-\left(\frac{\partial U_s}{\partial x_1}+\frac{\partial V_s}{\partial y}+\frac{\partial W_s}{\partial z_1}\right)\left(\gamma M_\infty^2\zeta_4-\frac{\zeta_5}{T_s}\right)$$

$$-V_s\left(\gamma M_\infty^2\frac{\partial\zeta_4}{\partial y}-\frac{\gamma M_\infty^2\zeta_4}{T_s}\frac{\partial T_s}{\partial y}-\frac{1}{T_s}\frac{\partial\zeta_5}{\partial y}+\frac{2\zeta_5}{T_s^2}\frac{\partial T_s}{\partial y}\right)$$

$$-\gamma M_\infty^2 W_s\left(\frac{\partial\zeta_4}{\partial z_1}-\frac{\zeta_4}{T_s}\frac{\partial T_s}{\partial z_1}\right)+\frac{\zeta_7}{T_s}\frac{\partial T_s}{\partial z_1}+\frac{W_s}{T_s}\frac{\partial\zeta_5}{\partial z_1}$$

$$-\frac{2W_s\zeta_5}{T_S^2}\frac{\partial T_S}{\partial z_1}-\frac{\partial\zeta_7}{\partial z_1}+O(R^{-1})$$

$$G_4=-\frac{1}{T_s}\left[U_s\frac{\partial\zeta_3}{\partial x_1}+V_s\frac{\partial\zeta_3}{\partial y}+\zeta_3\frac{\partial V_s}{\partial y}\right.$$

$$\left.+W_s\frac{\partial\zeta_3}{\partial z_1}\right]+O(R^{-1})$$

$$G_6=-\frac{RPr}{\mu_s}\left[-\frac{U_s}{T_s}\frac{\partial\zeta_5}{\partial x_1}-\frac{V_s}{T_s}\frac{\partial\zeta_5}{\partial y}-\frac{W_s}{T_s}\frac{\partial\zeta_5}{\partial z_1}\right.$$

$$-\left(\frac{\gamma M_\infty^2\zeta_4}{T_s}-\frac{\zeta_5}{T_s^2}\right)\left(U_s\frac{\partial T_s}{\partial x_1}+V_s\frac{\partial T_s}{\partial y}+W_s\frac{\partial T_s}{\partial z_1}\right)$$

$$+(\gamma-1)M_\infty^2\left(\frac{\partial P_s}{\partial x_1}\zeta_1+U_s\frac{\partial\zeta_4}{\partial x_1}+V_s\frac{\partial\zeta_4}{\partial y}\right.$$

$$\left.-\frac{\zeta_1}{T_s}\frac{\partial T_s}{\partial x_1}-\frac{\zeta_7}{T_s}\frac{\partial T_s}{\partial z_1}+\frac{\partial P_s}{\partial z_1}\zeta_7+W_s\frac{\partial\zeta_4}{\partial z_1}\right]+O(1)$$

$$G_8=\frac{R}{\mu_s}\left[\frac{U_s}{T_s}\frac{\partial\zeta_7}{\partial x_1}+\frac{\zeta_1}{T_s}\frac{\partial W_s}{\partial x_1}+\frac{V_s}{T_s}\frac{\partial\zeta_7}{\partial y}+\frac{W_s}{T_s}\frac{\partial\zeta_7}{\partial z_1}\right.$$

$$+\frac{\zeta_7}{T_s}\frac{\partial W_s}{\partial z_1}+\frac{\partial\zeta_4}{\partial z_1}+\left(\frac{\gamma M_\infty^2\zeta_4}{T_s}-\frac{\zeta_5}{T_s^2}\right)\left(U_s\frac{\partial W_s}{\partial x_1}\right.$$

$$\left.\left.+V_s\frac{\partial W_s}{\partial y}+W_s\frac{\partial W_s}{\partial z_1}\right)\right]+O(1)$$

Appendix C

$$g_1=i\sum_{m,n=1}^{8}\int_0^\infty\frac{\partial a_{nm}}{\partial\omega}\zeta_m\zeta_n^*\mathrm{d}y$$

$$g_2=-i\sum_{m,n=1}^{8}\int_0^\infty\frac{\partial a_{nm}}{\partial\alpha}\zeta_m\zeta_n^*\mathrm{d}y$$

$$g_3=-i\sum_{m,n=1}^{8}\int_0^\infty\frac{\partial a_{nm}}{\partial\beta}\zeta_m\zeta_n^*\mathrm{d}y$$

$$\omega_\alpha=g_2/g_1,\qquad \omega_\beta=g_3/g_1$$

$$h_1=-g_1^{-1}\sum_{m=1}^{8}\int_0^\infty G_m\zeta_m^*\mathrm{d}y$$

$$h_2=ig_1^{-1}\sum_{m=1}^{8}\int_0^\infty\frac{\partial a_{nm}}{\partial x_1}\bigg|_{\alpha,\beta}\zeta_m\zeta_n^*\mathrm{d}y$$

$$h_3=ig_1^{-1}\sum_{m,n=1}^{8}\int_0^\infty\frac{\partial a_{nm}}{\partial z_1}\bigg|_{\alpha,\beta}\zeta_m\zeta_n^*\mathrm{d}y$$

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